Darcy’s Law, Richards’ Equation, and Green-Ampt Equation

1. Darcy’s Law

Fluid potential: in classic hydraulics, the fluid potential $\Phi$ is stated in terms of Bernoulli Equation

$$P, \text{ pressure, } [F \text{ L}^{-2}]$$
$$\gamma, \text{ weight density of fluid, } [F \text{ L}^{-3}]$$
$$g, \text{ gravitational acceleration, } [L \text{ T}^{-2}]$$
$$z, \text{ elevation above a reference level, } [L]$$
$$V, \text{ velocity, } [L \text{ T}^{-1}]$$

The three terms represent the pressure, gravity, and velocity potentials, respectively. As flow in soil is very slow, $V \approx 0$. If $\rho$ and $g$ can be assumed constant and $p_0 = 0$ then Eq 1.1 becomes

$$Darcy's \ law$$

$$V_x, \text{ volumetric flow rate in the } x \text{ direction per unit cross-sectional area, } [LT^{-1}]$$
$$K, \text{ hydraulic conductivity at saturation } [LT^{-1}]$$
$$z, \text{ elevation above an arbitrary datum } [L]$$
$$p, \text{ water pressure } [FL^{-2}]$$
$$\gamma_w, \text{ weight density of water } [FL^{-3}]$$

For unsaturated flow, let $\psi = \frac{P}{\gamma_w}$, as both $K$ and $\psi$ are a function of soil water content $\theta$ (see Fig. 1 for typical relationships of $\psi$ vs $\theta$ and $K(\theta)$ vs $\theta$), Eq 1.3 is often written as

$$2. Richards’ Equation$$

- Richards equation is the basic theoretical equation for vertical unsaturated flow.
- The equation is not applicable to macropore flows.
- Analytical solutions are available with certain boundary and initial conditions and assumptions. Numerical solutions can be obtained for more flexible conditions by using the finite difference or finite element methods.
Fig. 1. Water release characteristic curve ($\psi$ vs $\theta$) and unsaturated hydraulic conductivity as a function of soil water ($K(\theta)$ vs $\theta$).
The solutions to the equation, analytical or numerical, would present a relation between soil water content, spatial location, and time.

Derivation based on Dingman (2002)

**Domain**: a rectangular parallelepiped of soil is oriented so that one dimension is aligned with the vertical $z$ direction of a rectangular coordinate system (see Fig. 2). $\Delta x$, $\Delta y$, $\Delta z$ are “small” but large enough to encompass a representative volume of the soil. Flow occurs only vertically downward ($z'$ direction).

During a “small” $\Delta t$, the mass entering in the volume is

$$V_z N \Delta t$$

(2.1)

where $V_z$ is the flow velocity.

The mass leaving the volume is

$$\rho_0 \Delta x \Delta y \Delta z$$

(2.2)

The change of mass within the volume is

$$\frac{\partial \theta}{\partial z'} = -1$$

(2.3)

where $\theta$ is the volumetric soil water content.

Conservation of mass law requires

$$(2.1) - (2.2) = (2.3), \text{ and therefore}$$

(2.4)

Assuming constant $\rho_0$ and simplifying Eq 2.4 leads to

(2.5)

Applying Darcy’s law

(2.6)

Since $\frac{\partial \theta}{\partial z'} = -1$ so Eq 2.6 becomes
Fig. 2. Schematic of REV and terms used in deriving the Richards equation (from Dingman, 2002).
Taking the derivative with respect to $z'$ in Eq 2.7 yields

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z} \left[ K(\psi) \left( \frac{\partial \psi}{\partial z} + 1 \right) \right]$$  \hspace{1cm} (2.10)

where $C_w(\psi) = \frac{\partial \theta}{\partial \psi}$ is defined as the water capacity function, and $K(\psi)$ is another form of the unsaturated hydraulic conductivity function.

Both forms of Richards’ equation are used in unsaturated-zone modeling because each has unique advantages and disadvantages in numerical implementation. The $\theta$-based form enables a fast solution in terms of computer CPU time and conserves mass balance, but cannot be used to simulate flow under saturated conditions. The $\psi$-based form is more suitable for variably-saturated soil (including fully saturated) conditions and spatially heterogeneous soils. However, numerical problems occur when water infiltrates into an initially dry soil, as the steep gradients in the water potential at the wetting front tend to cause unstable solutions. The reason for this numerical instability lies in the strong nonlinearity of the soil moisture characteristic and unsaturated hydraulic conductivity functions. Several improvements of the numerical solution for the Richards’ equation have been proposed. The use of a mixed-form of Richards’ equation in combination with a modified Picard iteration conserve mass balance without requiring much additional computational effort. The mixed-form of the Richards’ equation still needs small time steps when steep potential gradients occur, though. To overcome this limitation, variable transformations were introduced to make the numerical problem less nonlinear, and to make the solution more robust.

3. Green-Ampt Equation

- based on Darcy’s law and the principle of conservation of mass for idealized conditions
- in a finite-difference formulation and simpler than the numerical solutions of the Richards’ equation

Domain: a block of soil homogeneous to an indefinite depth, i.e., porosity $\phi$ and saturated hydraulic conductivity $K_s$ are invariant throughout the domain. In addition, no water table, capillary fringe, or impermeable layer is present in the domain.
Boundary condition: assuming no ET and no ponding.

Initial condition: water content prior to $t = 0$ is invariant at an initial value of $\Theta_0 < \phi$

Other symbols:
- $z$: vertical axis, [L]
- $z'$: downward direction, [L]
- $f(t)$: infiltration rate at time $t$, [L T$^{-1}$]
- $F(t)$: total amount of water infiltrated up to time $t$, [L]
- $V_z$: downward water flux [L T$^{-1}$]
- $\Theta$: volumetric soil water content
- $K(\Theta)$: unsaturated hydraulic conductivity function, [L T$^{-1}$]
- $\psi(\Theta)$: matric potential function, [L]
- $\Theta_{fc}$: soil water content at field capacity
- $w$: water input rate, constant, [L T$^{-1}$]
- $t_w$: time during which water input occurs, [T]
- $t_p$: the instant when the soil surface layer becomes saturated, [T]

Flux before water input begins: just before $t = 0$, the downward flux of water $V_z(z, 0)$ is given by

$$V_z(z, 0)$$

from Darcy’s law and due to the fact that no vertical water content or matric potential gradient exists.

This is not a steady-state as the soil is gradually draining, but if $\Theta_0 \leq \Theta_{fc}$, $V_z(z, 0)$ is negligible.

Case 1: water input rate less than saturated hydraulic conductivity ($w < K_s$)

Consider a thin surface layer of soil where Eq 3.1 applies at the instant water input begins. If $w > K(\Theta_0)$, water will enter this layer faster than it is leaving. The excess then goes into storage, increasing the layer’s water content $\Theta$. The increase in $\Theta$ causes an increase in $K(\Theta)$, and in turn an increase in downward flux. However, as long as the water content in this layer is less than the water content at which the hydraulic conductivity equals the water input rate, i.e., $\Theta \leq \Theta_w$ and $K(\Theta) < w$, the water content will continue to increase until $\Theta = \Theta_w$ and $K(\Theta_w) = w$ when outflow equals inflow and there is no further change in $\Theta$ till the end of $t_w$.

Such a process happens successively in each layer as $w$ continues, resulting in a successive $\Theta$ profile and $f(t)$ functions.

Case 2: water input rate greater than saturated hydraulic conductivity ($w > K_s$)

When $w > K_s$, the aforementioned process will occur in the early stage of infiltration. Water will arrive at each layer faster than can be transmitted and will initially go into storage, increasing $\Theta$ and $K(\Theta)$. However, $\Theta$ cannot exceed $\phi$, and $K(\Theta)$ cannot increase beyond $K_s$. After the surface reaches saturation, some input water will continue to infiltrate, but the excess will accumulate on the surface as ponding.
We can develop an equation for $t_p$, if assuming the wetting front as a perfectly sharp boundary, which is at a depth $z_f(t_p)$. Up to this point, all the water input has infiltrated, so

$$F(t_p) = wt_p \quad (3.2)$$

All this water occupies the soil between the surface and $z_f(t_p)$, therefore

$$\text{(3.3)}$$

Combining Eq 3.2 and 3.3 yields

$$\text{(3.4)}$$

In order to use Eq 3.4, we need to determine $z_f(t_p)$ by applying Darcy’s law (Eq 2.7) in finite-difference form between the surface and the depth $z_f(t_p)$

$$\text{(3.5)}$$

where $\psi_f$ is the effective tension at the wetting front. This relation is justified as at $t = t_p$, saturation occurs at the soil surface so the matric potential is 0, the hydraulic conductivity is $K_s$ and the infiltration rate is just equal to the rainfall rate. Note that $\psi_f < 0$, we can solve Eq 3.5 for $z_f(t_p)$

$$\text{(3.6)}$$

Substituting Eq 3.6 into Eq 3.4 gives

$$\text{(3.7)}$$

This equation has a logical form in that $t_p$ increases with increasing $K_s$, $|\psi_f|$, and the initial soil water deficit, and decreases with increasing $w$.

As water input continues after $t = t_p$, infiltration continues, but at a decreasing rate. Below we will develop a solution of infiltration for this phase. For time $t_p < t < t_w$, rewriting Eq 3.5 and accounting for a depth of ponding $y(t)$ leads to

$$\text{(3.8)}$$

Note that $y(t)$ is generally a complex function that depends on the amount of infiltration up to time $t$ as well as the surface slope and roughness conditions. Previous studies have shown that satisfactory results can be obtained by assuming negligible $y(t)$. Hence, we will also assume $y(t) = 0$. 

7
Continuity requires that

\[ (3.9) \]

Solving for \( z(t) \) and substituting into Eq 3.8 yields

\[ \text{for } t_p \leq t \leq t_w \]  \hspace{1cm} (3.10)

Eq 3.10 allows us to compute the infiltration rate, which is now the infiltration capacity, as a function of the total infiltration that has occurred.