Equation (3.65) is called the soil water conservation or continuity equation. If we had allowed the water to flow in an arbitrary direction in Fig. 3.19, the water conservation equation would be written as

$$\frac{\partial J_{w_x}}{\partial t} + \frac{\partial J_{w_y}}{\partial y} + \frac{\partial J_{w_z}}{\partial z} + \frac{\partial \theta}{\partial t} + r_w = 0 \quad (3.66)$$

where $J_{w_x}, J_{w_y}, J_{w_z}$ are the components of the water flux vector

$$J_w = J_{w_x} \hat{i} + J_{w_y} \hat{j} + J_{w_z} \hat{k} \quad (3.67)$$

and \( \hat{i}, \hat{j}, \hat{k} \) are unit vectors in the \( x, y, z \) directions, respectively.

The water uptake term \( r_w \) is included for completeness in (3.65) and (3.66). It is equal to zero when there are no plant roots or other sinks of water present and must be specified with a model when water uptake is occurring. Processes involving water uptake will be discussed later in the book.

### 3.3.6 Richards Equation for Transient Water Flow

The water conservation equation relates water fluxes, storage changes, and sources or sinks of water. When it is combined with the Buckingham–Darcy flux equation (3.27), an equation may be derived to predict the water content or matric potential in soil during transient flow. We will assume for simplicity that the flow is vertical and that no plant roots are present \( (r_w = 0) \).

Inserting (3.27) into (3.65) produces

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[ K(h) \left( \frac{\partial h}{\partial z} + 1 \right) \right] \quad (3.68)$$

Equation (3.68) may not be solved in the form it is in, because it contains two unknowns \( \theta \) and \( h \) and only one equation. This difficulty may be overcome by using the water characteristic or matric potential–water content function \( h(\theta) \) to eliminate either \( \theta \) or \( h \) from (3.68). Since either variable may be eliminated, there are two forms of the equation.

**Water Content Form of Richards Equation** The flux equation (3.27) may be reexpressed as a function of \( \theta \) alone by the following transformations:

1. Since \( K(h) \) is a function of \( h \) and \( h(\theta) \) is a function of \( \theta \), \( K \) may be written directly as a function of \( \theta \):

$$K(h(\theta)) = K(\theta) \quad (3.69)$$
2. The partial derivative \( \partial h / \partial z \) may be rewritten by the chain rule of differentiation (Kaplan, 1984) as

\[
\frac{\partial h(\theta)}{\partial z} = \frac{dh}{d\theta} \frac{\partial \theta}{\partial z}
\]  

(3.70)

where \( dh/d\theta \) is the slope of the matric potential–water content function.

Inserting (3.69) and (3.70) into (3.27) produces

\[
J_w = -K(\theta) \frac{dh}{d\theta} \frac{\partial \theta}{\partial z} - K(\theta) = -D_w(\theta) \frac{\partial \theta}{\partial z} - K(\theta)
\]  

(3.71)

where

\[
D_w(\theta) = K(\theta) \frac{dh}{d\theta}
\]  

(3.72)

is called the soil water diffusivity.

After this transformation, (3.68) may be written as

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( D_w(\theta) \frac{\partial \theta}{\partial z} \right) + \frac{\partial K(\theta)}{\partial z}
\]  

(3.73)

Equation (3.73) is called the water content form of the Richards equation. It is a second-order, nonlinear, partial differential equation called a Fokker–Planck equation and can generally only be solved by numerical methods. Assum ing that \( D_w(\theta) \) and \( K(\theta) \) are known, (3.73) requires two boundary conditions (i.e., the soil surface and deep in the soil) where the behavior of \( \theta \) or \( J_w \) is known as a function of time. It also requires specification of an initial condition describing the water content of the entire profile at \( t = 0 \).

Modern high-speed computers have made the task of solving (3.73) almost routine when the soil water functions and boundary conditions are known. Use of this equation as a predictive tool is limited primarily by difficulties in measuring \( D_w \) and \( K \) accurately over the region where the simulation is to be run, particularly when the soil is heterogeneous and each location in the soil has a different \( K(\theta) \) and \( D_w(\theta) \) function.

Equation (3.73) has also been derived by ignoring hysteresis. The slope \( dh/d\theta \) is defined only for a uniform wetting or drying process in which \( h(\theta) \) is described uniquely by a single curve (see Fig. 3.12). When repeated wetting and drying cycles are present, (3.73) is invalid and the water diffusivity \( D_w \) cannot even be defined. Only a limited amount of study of the influence of hysteresis on water flow has been performed, either theoretically or experimentally, although there are some indications that it may have a substantial influence in certain cases (Curtis and Watson, 1984; Jones and Watson, 1987; Russo et al., 1989).